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Three-tangle does not properly quantify tripartite entanglement for Greenberger-Horne-Zeilinger-type states

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Some mixed states composed of only Greenberger-Horne-Zeilinger (GHZ) states can be expressed in terms of only W states. This fact implies that such states have vanishing three-tangle. One of such rank-3 states, Π_{GHZ}, is explicitly presented in this Rapid Communication. These results are used to compute analytically the three-tangle of a rank-4 mixed state σ composed of four GHZ states. This analysis with considering Bloch sphere S^{16} of $d=4$ qudit system allows us to derive the hyperpolyhedron. It is shown that the states in this hyperpolyhedron have vanishing three-tangle. Computing the one-tangles for Π_{GHZ} and σ, we prove the monogamy inequality explicitly. Making use of the fact that the three-tangle of Π_{GHZ} is zero, we try to explain why the W class in the whole mixed states is not of measure zero contrary to the case of pure states.

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Nowadays, it is well known that entanglement is the most valuable physical resource for quantum information processing such as quantum teleportation [1], superdense coding [2], quantum cloning [3], quantum algorithms [4], quantum cryptography [5], and quantum computer technology [6]. Thus, it is highly important to understand the various properties of the multipartite entanglement of the quantum states.

The main obstacle for characterizing the entanglement of the multipartite state is its calculational difficulties even if original definition of the entanglement measure itself is comparatively simple. In addition, computation of the entanglement for the multipartite mixed states is much more difficult than that for the pure states mainly due to the fact that the entanglement for the mixed states, in general, is defined by a convex-roof extension [7]. In order to compute the entanglement explicitly for the mixed states, therefore, we should find an optimal decomposition of the given mixed state, which provides a minimum value of the entanglement over all possible ensembles of pure states. However, there is no general way for finding the optimal decomposition for the arbitrary mixed states except bipartite cases [8]. Thus, it becomes a central issue for the computation of the mixed state entanglement.

A few years ago, fortunately, Wootters [8] showed how to construct the optimal decompositions for the most simple bipartite cases. This enables us to compute the concurrence, one of the entanglement measures, analytically for the arbitrary two-qubit mixed states. It also makes it possible to understand more deeply the role of the entanglement in the real quantum information processing [9]. Most importantly, it becomes a basis for the quantification of three-party entanglement called residual entanglement or three-tangle [10]. Thus, it is extremely important to find a calculation tool for the three-tangle if one wants to take a step toward a fundamental issue, i.e., characterization of the multipartite mixed state entanglement.

It is well known [11] that the three-qubit pure states can be classified by product states ($A-B-C$), biseparable states ($A-BC, B-AC, C-AB$), and true tripartite states (ABC) through stochastic local operation and classical communication. The true tripartite states consist of two different classes, Greenberger-Horne-Zeilinger (GHZ) class and W class, where

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{1}$$

Since the three-tangle τ_3 for the pure state $|\psi\rangle = \Sigma_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$ is defined as [10]

$$\tau_3 = 4|d_1 - 2d_2 + 4d_3|, \tag{2}$$

with

$$d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2,$$

$$d_2 = a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001},$$

$$d_3 = a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}, \tag{3}$$

it is easy to show that the product and biseparable states have zero three-tangle. This fact implies that the three-tangle is a genuine measure for the three-party entanglement.

However, there is a crucial defect in the three-tangle as a three-party entanglement measure. While the three-tangle for the GHZ state is maximal, i.e., $\tau_3(\mathrm{GHZ})=1$, it vanishes for the W state. This means that the three-tangle does not properly quantify the three-party entanglement for the W-type states. The purpose of this Rapid Communication is to show that besides W-type states the three-tangle τ_3 does not properly quantify the three-party entanglement for rank-3 mixtures composed of only three GHZ-type states.

Recently, the three-tangle for rank-2 mixture of GHZ and W states is analytically computed [12]. In Ref. [13], furthermore, the three-tangle for the rank-3 mixture of GHZ, W, and inverted W states is also analytically computed. In this Rapid

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Communication we start with showing that a mixed state

$$\Pi_{GHZ} = \frac{1}{3}[|\mathrm{GHZ},2+\rangle\langle\mathrm{GHZ},2+| + |\mathrm{GHZ},3+\rangle\langle\mathrm{GHZ},3+| + |\mathrm{GHZ},4+\rangle\langle\mathrm{GHZ},4+|] \tag{4}$$

has vanishing three-tangle, which we define for later use as the following:

$$\begin{aligned}|\mathrm{GHZ},1\pm\rangle &= \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle),\\ |\mathrm{GHZ},2\pm\rangle &= \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle),\\ |\mathrm{GHZ},3\pm\rangle &= \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle),\\ |\mathrm{GHZ},4\pm\rangle &= \frac{1}{\sqrt{2}}(|011\rangle \pm |100\rangle).\end{aligned} \tag{5}$$

Let us consider a pure state

$$|J(\theta_1,\theta_2)\rangle = \frac{1}{\sqrt{3}}|\mathrm{GHZ},2+\rangle - \frac{1}{\sqrt{3}}e^{i\theta_1}|\mathrm{GHZ},3+\rangle - \frac{1}{\sqrt{3}}e^{i\theta_2}|\mathrm{GHZ},4+\rangle. \tag{6}$$

Then, it is easy to show that the three-tangle of $|J(\theta_1,\theta_2)\rangle$ is

$$\tau_3(\theta_1,\theta_2) = \frac{1}{9}|1-(e^{i\theta_1}-e^{i\theta_2})^2||1-(e^{i\theta_1}+e^{i\theta_2})^2|, \tag{7}$$

which vanishes when

$$(\theta_1,\theta_2) = \left\{\begin{matrix}(\pi/3,2\pi/3),(5\pi/3,4\pi/3)\\ (2\pi/3,\pi/3),(4\pi/3,5\pi/3)\\ (\pi/3,5\pi/3),(5\pi/3,\pi/3)\\ (2\pi/3,4\pi/3),(4\pi/3,2\pi/3)\end{matrix}\right\}. \tag{8}$$

Moreover, one can show straightforwardly that Π_{GHZ} can be decomposed into

$$\begin{aligned}\Pi_{GHZ} = \frac{1}{8}[&|J(\pi/3,2\pi/3)\rangle\langle J(\pi/3,2\pi/3)|\\ &+|J(5\pi/3,4\pi/3)\rangle\langle J(5\pi/3,4\pi/3)|\\ &+|J(\pi/3,5\pi/3)\rangle\langle J(\pi/3,5\pi/3)|\\ &+|J(2\pi/3,4\pi/3)\rangle\langle J(2\pi/3,4\pi/3)|\\ &+\text{terms with exchanged arguments}].\end{aligned} \tag{9}$$

Combining Eqs. (8) and (9), one can show that Eq. (9) is the optimal decomposition of Π_{GHZ} and its three-tangle is zero:

$$\tau_3(\Pi_{GHZ}) = 0. \tag{10}$$

The reason why Π_{GHZ} has vanishing three-tangle is that the optimal ensembles given in Eq. (9) are all W states. Therefore, Π_{GHZ} can also be expressed in terms of only W states. As a result, we encounter a very strange situation where Π_{GHZ} has vanishing three- and two-tangles,[1] but nonvanishing one-tangle

$$4\min[\det(\mathrm{Tr}_{BC}\Pi_{GHZ})] = \frac{5}{9}. \tag{11}$$

For comparison one can compute π-tangle [14], another three-party entanglement measure defined in terms of the global negativities [15]. It is easy to show that the π-tangle of Π_{GHZ} is not vanishing but 1/9. This fact seems to indicate that the three-tangle does not properly reflect the three-party entanglement for GHZ-type states as well as W-type states.

We can use Eq. (10) for computing the three-tangles of the higher-rank mixed states. For example, let us consider the following rank-4 state:

$$\sigma = x|\mathrm{GHZ},1+\rangle\langle\mathrm{GHZ},1+| + (1-x)\Pi_{GHZ} \tag{12}$$

with $0 \le x \le 1$. In order to compute the three-tangles for σ we first consider a pure state

$$|X(x,\varphi_1,\varphi_2,\varphi_3)\rangle = \sqrt{x}|\mathrm{GHZ},1+\rangle - \sqrt{\frac{1-x}{3}}(e^{i\varphi_1}|\mathrm{GHZ},2+\rangle + e^{i\varphi_2}|\mathrm{GHZ},3+\rangle + e^{i\varphi_3}|\mathrm{GHZ},4+\rangle). \tag{13}$$

Then it is easy to show that the three-tangle of $|X(x,\varphi_1,\varphi_2,\varphi_3)\rangle$ becomes

$$\begin{aligned}\tau_3(|X(x,\varphi_1,\varphi_2,\varphi_3)\rangle) = \Bigg| & x^2 + \frac{(1-x)^2}{9}(e^{4i\varphi_1}+e^{4i\varphi_2}+e^{4i\varphi_3})\\ &- \frac{2}{3}x(1-x)(e^{2i\varphi_1}+e^{2i\varphi_2}+e^{2i\varphi_3}) - \frac{2}{9}(1-x)^2(e^{2i(\varphi_1+\varphi_2)}\\ &+ e^{2i(\varphi_1+\varphi_3)} + e^{2i(\varphi_2+\varphi_3)}) - \frac{8\sqrt{3}}{9}\sqrt{x(1-x)^3}e^{i(\varphi_1+\varphi_2+\varphi_3)}\Bigg|.\end{aligned} \tag{14}$$

The vectors $|X(x,\varphi_1,\varphi_2,\varphi_3)\rangle$ have the following properties. The three-tangle of it has the largest zero at $x=x_0 \equiv 3/4$ and $\varphi_1=\varphi_2=\varphi_3=0$. The vectors $|X(x,0,0,0)\rangle$, $|X(x,0,\pi,\pi)\rangle$, $|X(x,\pi,0,\pi)\rangle$, and $|X(x,\pi,\pi,0)\rangle$ have same three-tangles. Finally, σ can be decomposed into

$$\begin{aligned}\sigma = \frac{1}{4}[&|X(x,0,0,0)\rangle\langle X(x,0,0,0)| + |X(x,0,\pi,\pi)\rangle\langle X(x,0,\pi,\pi)|\\ &+|X(x,\pi,0,\pi)\rangle\langle X(x,\pi,0,\pi)|\\ &+|X(x,\pi,\pi,0)\rangle\langle X(x,\pi,\pi,0)|].\end{aligned} \tag{15}$$

When $x \le x_0$, one can construct the optimal decomposition in the following form:

[1] It is easy to show that $\mathcal{C}_{AB}^2$ and $\mathcal{C}_{AC}^2$ are zero, where $\mathcal{C}$ is concurrence for corresponding reduced states.

$$\sigma = \frac{x}{4x_0}[|X(x_0,0,0,0)\rangle\langle X(x_0,0,0,0)| + |X(x_0,0,\pi,\pi)\rangle \times\langle X(x_0,0,\pi,\pi)| + |X(x_0,\pi,0,\pi)\rangle\langle X(x_0,\pi,0,\pi)| + |X(x_0,\pi,\pi,0)\rangle\langle X(x_0,\pi,\pi,0)|] + \frac{x_0-x}{x_0}\Pi_{GHZ}. \quad (16)$$

Since Π_{GHZ} has the vanishing three-tangle, one can show easily

$$\tau_3(\sigma) = 0 \text{ when } x \leq x_0 = 3/4. \quad (17)$$

Now, let us consider the three-tangle of σ in the region $x_0 \leq x \leq 1$. Since Eq. (15) is an optimal decomposition at $x = x_0$, one can conjecture that it is also optimal in the region $x_0 \leq x$. As will be shown shortly, however, this is not true at the large-x region. If we compute the three-tangle under the condition that Eq. (15) is optimal at $x_0 \leq x$, its expression becomes

$$\alpha_I(x) = x^2 - \frac{1}{3}(1-x)^2 - 2x(1-x) - \frac{8\sqrt{3}}{9}\sqrt{x(1-x)^3}. \quad (18)$$

However, one can show straightforwardly that $\alpha_I(x)$ is not a convex function in the region $x \geq x_*$, where

$$x_* = \frac{1}{4}(1 + 2^{1/3} + 4^{1/3}) \approx 0.961\,831. \quad (19)$$

Therefore, we need to convexify $\alpha_I(x)$ in the region $x_1 \leq x \leq 1$ to make the three-tangle to be convex function, where x_1 is some number between x_0 and x_*. The number x_1 will be determined shortly.

In the large-x region one can derive the optimal decomposition in a form

$$\sigma = \frac{1-x}{4(1-x_1)}[|X(x_1,0,0,0)\rangle\langle X(x_1,0,0,0)| + |X(x_1,0,\pi,\pi)\rangle \times\langle X(x_1,0,\pi,\pi)| + |X(x_1,\pi,0,\pi)\rangle\langle X(x_1,\pi,0,\pi)| + |X(x_1,\pi,\pi,0)\rangle\langle X(x_1,\pi,\pi,0)|] + \frac{x-x_1}{1-x_1}|\text{GHZ},1+\rangle \times\langle \text{GHZ},1+|, \quad (20)$$

which gives a three-tangle as

$$\alpha_{II}(x,x_1) = \frac{1-x}{1-x_1}\alpha_I(x_1) + \frac{x-x_1}{1-x_1}. \quad (21)$$

Since $d^2\alpha_{II}/dx^2 = 0$, there is no convex problem if $\alpha_{II}(x,x_1)$ is a three-tangle in the large-x region. The constant x_1 can be fixed from the condition of minimum α_{II}, i.e., $\partial\alpha_{II}(x,x_1)/\partial x_1 = 0$, which gives

$$x_1 = \frac{1}{4}(2+\sqrt{3}) \approx 0.933\,013. \quad (22)$$

As expected, x_1 is between x_0 and x_*. Thus, finally the three-tangle for σ becomes

$$\tau_3(\sigma) = \begin{cases} 0, & x \leq x_0 \\ \alpha_I(x), & x_0 \leq x \leq x_1 \\ \alpha_{II}(x,x_1), & x_1 \leq x \leq 1 \end{cases} \quad (23)$$

and the corresponding optimal decompositions are Eq. (16), Eq. (15), and Eq. (20) respectively. In order to show Eq. (23) is genuine optimal, first we plot x dependence of Eq. (14) for various φ_i $(i=1,2,3)$. These curves have been referred to as the characteristic curves [16]. Then, one can show, at least numerically, that Eq. (23) is a convex hull of the minimum of the characteristic curves, which implies that Eq. (23) is genuine three-tangle for σ.

It is straightforward to show that the mixture σ has vanishing two-tangles, i.e., $\mathcal{C}_{AB} = \mathcal{C}_{AC} = 0$, but nonvanishing one-tangle

$$\mathcal{C}^2_{A(BC)}(\sigma) = \frac{1}{9}[5 - 4x + 8x^2 - 8\sqrt{3x(1-x)^3}]. \quad (24)$$

Thus, the monogamy inequality $\tau_3 + \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} \leq \mathcal{C}^2_{A(BC)}$ holds for the rank-4 mixture σ.

Equation (10) can be used to compute the upper bound of the three-tangle for the higher-rank states. For example, let us consider the following rank-8 state:

$$\rho = \xi\sigma + (1-\xi)\tilde{\sigma} \quad (25)$$

where σ is given in Eq. (12) and $\tilde{\sigma}$ is

$$\tilde{\sigma} = y|\text{GHZ},1-\rangle\langle\text{GHZ},1-| + \frac{1-y}{3}[|\text{GHZ},2-\rangle\langle\text{GHZ},2-| + |\text{GHZ},3-\rangle\langle\text{GHZ},3-| + |\text{GHZ},4-\rangle\langle\text{GHZ},4-|]. \quad (26)$$

If $x = y$, σ and $\tilde{\sigma}$ are local-unitary (LU) equivalent with each other. Since the three-tangle is a LU-invariant quantity, $\tau_3(\tilde{\sigma})$ should be identical to $\tau_3(\sigma)$ when $x = y$.

Since ρ is a rank-8 mixed state, it seems to be extremely difficult to compute its three-tangle analytically. If, however, $0 \leq y \leq 3/4$, $\tau_3(\tilde{\sigma})$ becomes zero and the above analysis yields a nontrivial upper bound of $\tau_3(\rho)$ as follows:

$$\tau_3(\rho) \leq \xi\tau_3(\sigma). \quad (27)$$

In this Rapid Communication we have shown that the three-tangle does not properly quantify the three-party entanglement for some mixture composed of only GHZ states. This fact has been used to compute the (upper bound of) three-tangles for the higher-rank mixed states.

The fact $\tau_3(\sigma) = 0$ for $x \leq 3/4$ can be used to find other rank-4 mixtures that have vanishing three-tangle by considering the Bloch hypersphere of the $d=4$ qudit system. First, we correspond the GHZ states in σ to the basis of the qudit system as follows:

$$|\text{GHZ},1+\rangle = (1,0,0,0)^T, \quad |\text{GHZ},2+\rangle = (0,1,0,0)^T, \quad |\text{GHZ},3+\rangle = (0,0,1,0)^T, \quad |\text{GHZ},4+\rangle = (0,0,0,1)^T, \quad (28)$$

where T stands for transposition. It is well known [17] that the density matrix of the arbitrary $d=4$ qudit state can be represented by $\rho = (1/4)(I + \sqrt{6}\vec{n}\cdot\vec{\lambda})$, where $\vec{n}$ is a 15-dimensional unit vector and

$$\vec{\lambda} = (\Lambda_s^{12}, \ldots, \Lambda_s^{34}, \Lambda_a^{12}, \ldots, \Lambda_a^{34}, \Lambda^1, \Lambda^2, \Lambda^3). \tag{29}$$

The generalized Gell-Mann matrices Λ_s^{ij}, Λ_a^{ij}, and Λ^j are explicitly given in Ref. [17]. Then, the 15-dimensional Bloch vectors for $|X(3/4,0,0,0)\rangle$, $|X(3/4,0,\pi,\pi)\rangle$, $|X(3/4,\pi,0,\pi)\rangle$, and $|X(3/4,\pi,\pi,0)\rangle$ can be easily derived. Thus, these four points form a hyperpolyhedron in 16-dimensional space. Then all rank-4 quantum states corresponding to the points in this hyperpolyhedron have vanishing three-tangle.

As we have shown in this Rapid Communication, Π_{GHZ} has vanishing two- and three-tangle, but nonvanishing one-tangle. It makes the left-hand side of the monogamy inequality $\tau_3 + \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \le \mathcal{C}_{A(BC)}^2$ reduce to zero. Thus, the following natural question arises: what physical resources make the one-tangle to be nonvanishing? The authors in Ref. [18] conjectured that the origin of the nonvanishing one-tangle comes from the higher tangles of the purified state. To support their argument they considered a multipartite entanglement measure defined as

$$E_{ms}(\Psi_N) = \frac{\sum_k \tau_{k(R_k)} - 2\sum_{i<j} \mathcal{C}_{ij}^2}{N}, \tag{30}$$

where $\tau_{k(R_k)} = 2(1 - \mathrm{Tr}\,\rho_k^2)$ and $|\Psi_N\rangle$ is a N-qubit purified state of the given mixed state. Since the numerator of E_{ms} is the difference between the total one-tangle and total two-tangle, it measures a contribution of the higher-tangles to the one-tangle. If we choose the purified state as

$$|\Psi_5\rangle = \frac{1}{\sqrt{3}}|\mathrm{GHZ},2+\rangle|00\rangle + \frac{1}{\sqrt{3}}|\mathrm{GHZ},3+\rangle|01\rangle + \frac{1}{\sqrt{3}}|\mathrm{GHZ},4+\rangle \times |10\rangle, \tag{31}$$

$E_{ms}(\Psi_5)$ reduces to 43/45, which is larger than the one-tangle 5/9. Thus, it is possible that part of $E_{ms}(\Psi_5)$ converts into the nonvanishing one-tangle. However, still we do not know how to compute the one-tangle explicitly from $E_{ms}(\Psi_5)$.

The three-tangle itself is a good three-party entanglement measure. It exactly coincides with the modulus of a Cayley's hyperdeterminant [19] and is polynomial invariant under the local $SL(2,\mathbb{C})$ transformation [20]. As shown, however, it cannot properly quantify the three-party entanglement of W state and Π_{GHZ}: $\tau_3(W) = \tau_3(\Pi_{GHZ}) = 0$. On the other hand, the π-tangle gives the nonzero values: $\pi_3(W) = 4(\sqrt{5}-1)/9$ and $\pi_3(\Pi_{GHZ}) = 1/9$. Does this fact simply imply the crucial defects of the three-tangle as a three-party entanglement measure? Here, we would like to comment on the physical implication of $\tau_3(\Pi_{GHZ}) = 0$. A few years ago the three-qubit mixed states were classified in Ref. [21]. Following Ref. [21] the whole mixed states are classified as separable (S), biseparable (B), W, and GHZ classes. These classes satisfy $S \subset B \subset W \subset \mathrm{GHZ}$. One remarkable fact, which was proved in this reference, is that the $W \backslash B$ class is not of measure zero among all mixed states. This is contrary to the case of the pure states, where the set of W states forms measure zero [11]. This fact implies that the portion of $W \backslash B$ class in the whole mixed states becomes larger compared to that of W class in the whole pure states. How could this happen? The fact $\tau_3(\Pi_{GHZ}) = 0$ sheds light on this issue. Since Π_{GHZ} has zero three-tangle but nonzero π-tangle, it is manifestly an element of $W \backslash B$ class. As shown in Eq. (4), however, it consists of three GHZ states without pure W-type state. We think there are many $W \backslash B$ states, which are mixture of only GHZ states. It increases the portion of $W \backslash B$ class and eventually makes the $W \backslash B$ class to be of nonzero measure in the whole mixed states.

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